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REGULARLY OPEN SETS AND REGULAR SPACES

CHARLES DORSETT

Texas A & M University-Commerce
USA

Abstract

Within this paper recent discoveries concerning regularly open and regularly closed sets are used to further investigate and characterize regular spaces.

1. Introduction

Within this paper, all spaces are topological spaces. Regular spaces were introduced by Vietoris in 1921 [6].

Definition 1.1. A space (X, T) is regular iff for each $C \in C(T)$, the family of closed sets in (X, T) and each $x \notin C$, there exist disjoint open sets U and V such that $x \in U$ and $C \subseteq V$. A regular T_1 space is denoted by T_3.

Since 1921, regular spaces have been further investigated and characterized, not only providing greater understanding and possible uses of the regular property, but, also, leading to the discovery and resolution of related questions adding to the known properties of topological spaces. Today the regular separation axiom is included in the study of classical topology motivating and promoting their further investigation and characterization.

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In past studies, the semiregularization process proved to be useful in the study of regular spaces [7] suggesting that regularly open and regularly closed sets, and the semiregularization process could possibly be used to further investigate and characterize regular spaces. Within this paper, those possibilities are investigated.

Regularly open sets were introduced by Stone in 1937 [5].

Definition 1.3. Let (X, T) be a space and let $A \subseteq X$. Then A is regularly open, denoted by $A \in RO(X, T)$, iff $A = Int(Cl(A))$.

Within the 1937 paper [5], it was shown that the set of regularly open sets is a base for a topology Ts on X coarser that T, and the space (X, Ts) was called the *semiregularization space of* (X, T).

Following the introduction of regularly open sets was regularly closed sets [7].

Definition 1.4. Let (X, T) be a space and let $A \subseteq X$. Then A is regularly closed, denoted by $A \in RC(X, T)$, iff one of the following equivalent conditions is satisfied:

(a) $A = Cl(Int(A))$ or (b) $X \setminus A \in RO(X, T)$.

Within a recent paper [1], the question of what happens if the semiregularization process is repeated was resolved, showing that at most one new topology can be obtained by use of the semiregularization process. To accomplish the stated objective, it was proven that for a space (X, T), $RC(X, T) = \{Cl(O) | O \in T\}$ [1], which was combined with the fact that for a space (X, T), $RO(X, T) = \{Int(Cl(O)) | O \in T\}$ [2] to achieve the objective.

As indicated above, the semiregularization process has been useful in the continued investigation of regular spaces: "For a regular space (X, T), $T = Ts$ [7]. Thus the question of whether or not regularly open sets, regularly closed sets, and the semiregularization process can be used to gain additional insights into regular spaces naturally arises. Within this paper it is proven that, in fact, that is the case.

Given below are known results that will be used in the work below.

Theorem 1.1. *Let* (X, T) *be a space and let* $C(T)$ *denote the closed sets in* (X, T). *Then*

$$RO(X, T) = \{Int(Cl(O)) \mid O \in T\}[2]$$

$$= \{Ext(O) \mid O \in T\}[3]$$

$$= \{Int(C) \mid C \in C(T)\}[1],$$

and

$$RC(X, T) = \{Cl(Int(C)) \mid C \in C(T)\}$$

$$= \{Cl(U) \mid U \in RO(X, T)\}$$

$$= \{Cl(V) \mid V \in Ts\}[4].$$

Given below are additional results in the paper [4] that will be used in this paper.

Theorem 1.2. *Let* (X, T) *be a space, let* $C \in C(T)$, *and let* $D = Cl(Int(C))$. *Then* D *is regularly closed,* $Int(D) = Int(C)$, $Ext(Int(D)) = Ext(Int(C))$ *and* $Fr(Int(D)) = Fr(Int(C))$.

Theorem 1.3. *Let* (X, T) *be a space and let D and E be disjoint closed sets. Then*

$$X = Int(D) \cup Int(E) \cup Fr(Int(D \cup E)) \cup Ext(Int(D \cup E)),$$

where $Int(Fr(Int(D \cup E))) = \emptyset$.

2. New Characterizations of Regular Spaces

As known [7], if the space (X, T) is regular, then $T = Ts$. Simple examples can be given of spaces (X, T) for which $T = Ts$ and (X, T) is not regular. Also, examples can be given of a space (X, T) for which (X, Ts) is regular and (X, T) is not regular. Thus the semiregularization process alone is not sufficient to characterize regular spaces leading to questions concerning the use of regularly open sets or regularly closed sets. Below the question of what would be the circumstance if open in the definition of regular is replaced by regularly open is addressed and resolved.

Definition 2.1. A space (X, T) is regularly open regular iff for each $C \in C(T)$ and for each $x \notin C$, there exist disjoint regularly open sets U and V such that $x \in U$ and $C \subseteq V$.

Theorem 2.1. *Let (X, T) be a space. Then the following are equivalent*:

(a) (X, T) *is regular*,

(b) *for each $C \in C(T)$ and each $x \notin C$, there exists a closed set D such that $x \in Int(D)$ and $C \subseteq (X \setminus D)$*,

(c) *for each $C \in C(T)$ and each $x \notin C$, there exists a closed set D such that*

$$X = Int(D) \cup Fr(Int(D)) \cup Ext(Int(D)),$$

where

$$Int(Fr(Int(D))) = \phi, \ x \in Int(D) \ and \ C \subseteq Ext(Int(D)),$$

(d) *for each $C \in C(T)$ and each $x \notin C$, there exists a regularly closed set E such that*

$$X = Int(E) \cup Fr(Int(E)) \cup Ext(Int(E)),$$

where

$$x \in Int(E), \ C \subseteq Ext(Int(E)) \ and \ Int(Fr(Int(E))) = \phi,$$

(e) *for each $C \in C(T)$ and each $x \notin C$, there exists a regularly closed set E such that*

$$x \in Int(E) \text{ and } C \subseteq Ext(Int(E)),$$

(f) *for each $C \in C(T)$ and each $x \notin C$, there exists a regularly open set U such that $x \in U$ and $C \subseteq Ext(U)$*,

(g) (X, T) *is regularly open regular, and*

(h) *for each $C \in C(T)$ and each $x \notin C$, there exist disjoint Ts-open sets U and V such that $x \in U$* and $C \subseteq V$.

Proof. (a) implies (b): Let $C \in C(T)X$ and let $x \notin C$. Let U and V be disjoint open sets such that $x \in U$ and $C \subseteq V$. Then $D = Cl(U)$ is closed and $D \cap V = \varnothing$. Since $x \in U \subseteq Cl(U) = D$, then $x \in Int(D)$ and $C \subseteq V \subseteq (X \setminus D)$.

(b) implies (c): Let $C \in C(T)$ and let $x \notin C$. Let D be closed such that $x \in Int(D)$ and $C \subseteq (X \setminus D)$. The remainder of the proof is straightforward and is omitted.

(c) implies (d): Let $C \in C(T)$ and let $x \notin C$. Let D be closed such that

$$X = Int(D) \cup Fr(Int(D)) \cup Ext(Int(D)),$$

where $Int(Fr((Int(D)))) = \phi$, $x \in Int(D)$ and $C \subseteq Ext(Int(D))$. Let $E = Cl(Int(D))$. By Theorem 1.1, $E \in RC(X, T)$ and the remainder of the proof is straightforward and omitted.

Clearly (d) implies (e).

(e) implies (f): Let $C \in C(T)$ and let $x \notin C$. Let E be regularly closed such that $x \in Int(E)$ and $C \subseteq Ext(Int(E))$. Let $U = Int(E)$. Since E is regularly closed, then $E \in C(T)$ and, by Theorem 1.1, $U \in RO(X, T)$. Thus $x \in Int(E) = U$ and $C \subseteq Ext(Int(E)) = Ext(U)$.

(f) implies (g): Let $C \in C(T)$ and let $x \notin C$. Let U be regularly open such that $x \in U$ and $C \subseteq Ext(U)$. Since $RO(X, T) \subseteq T$, then, by Theorem 1.1, $Ext(U) \in RO(X, T)$ and $U \cap Ext(U) = \varnothing$. Hence (X, T) is regularly open regular.

Since $RO(X, T) \subseteq Ts$, (h) implies (g), and since $Ts \subseteq T$, (h) implies (a).

3. Additional Properties of Spaces and More Characterizations of Regular Spaces

Theorem 3.1. *Let (X, T) be a space and let D and E be disjoint elements in $C(T)$. Then*

$$Int(D \cup E) = Int(D) \cup Int(E),$$

$$Fr(Int(D \cup E)) = Fr(Int(D)) \cup Fr(Int(E))$$

and

$$Ext(Int(D \cup E)) = Ext(Int(D)) \cap Ext(Int(E)).$$

Proof. Since $(Int(D \cup E) \setminus E) \subseteq D$, then $(Int(D \cup E) \setminus E) \subseteq Int(D)$. Similarly $(Int(D \cup E) \setminus D) \subseteq Int(E)$. Thus

$$Int(D \cup E) = Int(D \cup E) \setminus (D \cap E)$$

$$= (Int(D \cup E) \setminus E) \cup (Int(D \cup E) \setminus D) \subseteq Int(D) \cup Int(E).$$

Since $(Int(D) \cup Int(E)) \subseteq Int(D \cup E)$, then $Int(D \cup E) = Int(D) \cup Int(D)$.

Let $x \in Fr(Int(D \cup E))$. Then $x \in Fr(Int(D) \cup Int(E))$ and $x \notin Int(D) \cup Int(E)$. Since

$$Fr(Int(D \cup E)) \subseteq Cl(Int(D \cup E))$$

$$= Cl(Int(D) \cup Int(E))$$

$$= Cl(Int(D)) \cup Cl(Int(E))$$

$$= Int(D) \cup Fr(Int(D)) \cup Int(E) \cup Fr(Int(E)),$$

then $Fr(Int(D \cup E)) \subseteq Fr(Int(D)) \cup Fr(Int(E))$. Let $y \in Fr(Int(D)) \cup Fr(Int(E))$. Then $y \notin Int(D) \cup Int(E) = Int(D \cup E)$. Since

$$Fr(Int(D)) \cup Fr(Int(E)) \subseteq Cl(Int(D)) \cup Cl(Int(E))$$

$$= Cl(Int(D) \cup Int(E))$$

$$= Cl(Int(D \cup E))$$

$$= Int((D \cup E)) \cup Fr(Int(D \cup E)),$$

then $y \in Fr(Int(D \cup E))$. Hence $Fr(Int(D)) \cup Fr(Int(E)) = Fr(Int(D \cup E))$.

By Theorem 1.3,

$$X = Int(D) \cup Int(Int(E)) \cup Fr(Int(D \cup E)) \cup Ext(Int(D \cup E)),$$

where $Int(Fr(Int(D \cup E))) = \varnothing$. Thus

$$X = Int(D) \cup Int(E) \cup Fr(Int(D)) \cup Fr(Int(E)) \cup Ext(Int(D \cup E))$$

$$= Cl(Int(D)) \cup Cl(Int(E)) \cup Ext(Int(D \cup E))$$

and

$$Ext(Int(D \cup E)) = X \setminus (Cl(Int(D)) \cup Cl(Int(E)))$$

$$= (X \setminus Cl(Int(D))) \cap (X \setminus Cl(Int(E)))$$

$$= Ext(Int(D)) \cap Ext(Int(E)).$$

Theorem 3.2. *Let (X, T) be a space. Then the following are equivalent*:

(a) (X, T) *is regular*,

(b) *for each $O \in T$ and each $x \in O$, there exist disjoint open sets U and V such that $x \in U \subseteq O$ and $Fr(O) \subseteq V$*,

(c) *for each $C \in C(T)$ and each $x \notin C$, there exist disjoint closed sets D and E such that $x \in Int(D)$ and $C \subseteq Int(E)$*,

(d) *for each $C \in C(T)$ and each $x \notin C$, there exist disjoint closed sets D and E such that*

$$X = Int(D) \cup Int(E) \cup Fr(Int(D \cup E)) \cup Ext(Int(D \cup E)),$$

where $x \in Int(D)$, $C \subseteq Int(E)$ and $Int(Fr(Int(D \cup E))) = \varnothing$,

(e) *for each $C \in C(T)$ and each $x \notin C$, there exist disjoint closed sets D and E such that*

$$X = Int(D) \cup Int(E) \cup Fr(Int(D)) \cup Fr(Int(E)) \cup (Ext(Int(D)) \cap Ext(Int(E))),$$

where $x \in Int(D)$, $C \subseteq Int(E)$ and $Int(Fr(Int(D))) = Int(Fr(Int(E))) = \phi$,

(f) *for each $C \in C(T)$ and each $x \notin C$, there exist disjoint regularly closed sets F and G such that*

$$X = Int(F) \cup Int(G) \cup Fr(Int(F)) \cup Fr(Int(G)) \cup (Ext(Int(F)) \cap Ext(Int(G))),$$

where $x \in Int(F)$, $C \subseteq Int(G)$ and $Int(Fr(Int(F))) = Int(Fr(Int(G))) = \phi$,

(g) *for each* $C \in C(T)$ *and each* $x \notin C$, *there exist disjoint regularly open sets U and V such that* $x \in U$ *and* $C \subseteq V$,

(h) *for each* $C \in C(T)$ *and each* $x \notin C$, *there exist disjoint Ts-open sets U and V such that* $x \in U$ *and* $C \subseteq V$,

(i) *for each* $O \in T$ *and each* $x \in O$, *there exist disjoint regularly open sets U and V such that* $x \in U$ *and* $Fr(O) \subseteq V$, *and*

(j) *for each* $O \in T$ *and each* $x \in O$, *there exist disjoint Ts-open sets U and V such that* $x \in U$ *and* $Fr(O) \subseteq V$.

Proof. Clearly (a) implies (b).

(b) implies (c): Let $C \in C(T)$ and $x \notin C$. Let $O = X \setminus C$. Then $x \in O \in T$. Let U and V be disjoint open sets such that $x \in U \subseteq O$ and $Fr(O) \subseteq V$. Let $E = Cl(V)$. Then E is closed, $U \cap E = \emptyset$ and $C \subseteq V \subseteq E$. Thus $C \subseteq Int(E)$. Since $x \in U \in T$, let W and Z be disjoint open sets such that $x \in W \subseteq U$ and $Fr(U) \subseteq Z$. Then

$$
\begin{aligned}
(X \setminus U) \cup Fr(U) &= Int(X \setminus U) \cup Fr(X \setminus U) \cup Fr(U) \\
&= Int(X \setminus U) \cup Fr(U) \subseteq Int(X \setminus U) \cup Z = Y \in T
\end{aligned}
$$

and $Y \cap W = \emptyset$. Let $D = Cl(W)$. Then $D \in C(T)$ such that $x \in Int(D)$ and $D \cap E = \emptyset$.

(c) implies (d): By Theorem 1.3, (c) implies (d). Combining (d) with Theorem 3.1 gives (e).

(e) implies (f): Let $C \in C(T)$ and let $x \notin C$. Let D and E be disjoint closed sets such that

$$X = Int(D) \cup Int(E) \cup Fr(Int(D)) \cup Fr(Int(E)) \cup (Ext(Int(D)) \cap Ext(Int(E))),$$

where $x \in Int(D)$, $C \subseteq Int(E)$ and $Int(Fr(Int(D))) = Int(Fr(Int(E))) = \phi$. Let $F = Cl(Int(D))$ and let $G = Cl(Int(E))$. Then, by Theorem 1.2, F and G are disjoint regularly closed sets. Thus combining the statement of (e) given above in this proof with Theorem 1.2 gives (f).

(f) implies (g): Let $C \in C(T)$ and let $x \notin C$. Let F and G be disjoint regularly closed sets such that

$$X = Int(F) \cup Int(G) \cup Fr(Int(F)) \cup Fr(Int(G)) \cup (Ext(Int(F)) \cap Ext(Int(G))),$$

where $x \in Int(F)$, $C \subseteq Int(G)$ and $Int(Fr(Int(F))) = Int(Fr(Int(G))) = \phi$. Since $RC(X, T) \subseteq C(T)$, then, by Theorem 1.1, $Int(F)$ and $Int(G)$ are disjoint regularly open sets such that $x \in Int(F)$ and $C \subseteq Int(G)$.

Since $RO(X, T) \subseteq Ts$, (g) implies (h).

(h) implies (i): Since $Ts \subseteq T$, then (X, T) is regular. Let $O \in T$ and let $x \in O$. Then $Fr(O) \in C(T)$ and $x \notin Fr(O)$, and by the arguments above, there exist disjoint regularly open sets U and V such that $x \in U$ and $Fr(O) \subset V$.

Since $RO(X, T) \subseteq Ts \subseteq T$, then (i) implies (j) and (j) implies (a).

Corollary 3.1. *Let (X, T) be a space. Then (X, T) is T_3 iff (X, T) is T_1 and for each $C \in C(T)$ and each $x \notin C$, there exists a closed sets D such that $x \in Int(D)$ and $C \subseteq X \setminus D$.*

In a similar manner, each of the characterizations of regular spaces can be extended to T_3 spaces.

Thus regularly open and regularly closed sets proved to be important in the continued investigation of regular spaces.

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